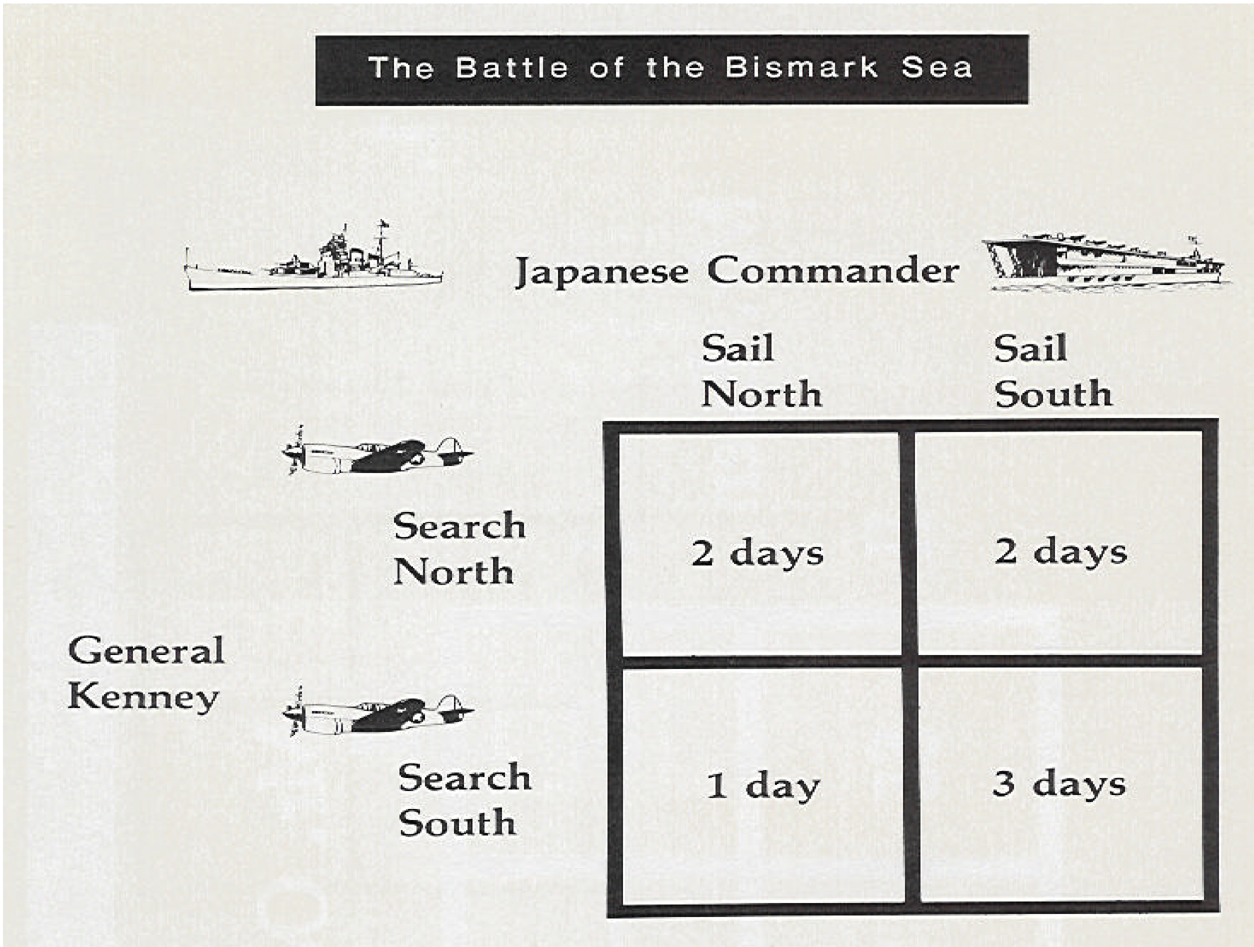
# Game Theory Agenda

- **Why are we doing this?**
  - **You as an analyst versus bean counter**
- **We will use two handouts to introduce the ideas**
- **An introduction to Game Theory, decision-making in competition against an intelligent opponent.**
  - **Basic set-up and some definitions**
  - **2x2 games with saddle points**
  - **Games with no saddle points**
  - **Solving 2x2 games with mixed strategies**
  - **Dominance**
  - **Solving 2xn and mx2 games**
  - **Solving general mxn games in general**

# Look at General Kenny's Dilemma




(1) Each side must choose an action without knowledge of the other's choice
(2) Both sides see the same payoff (or outcome) matrix
(3) One player's winnings are the other's losses (zero sum)

# Two ways to look at such games

- Game against nature
  - The opponent plays his strategy with some (known or estimated) probabilities
- Game against an intelligent opponent
  - He is doing the same type of planning you are. That is, his strategy depends on what he thinks your strategy might be.
- What type of game do you think fits most military situations?

# General Two-Player Zero-sum Game Set-up

- Where do we get:
  - Our alternatives?
  - Their alternatives?
  - The outcomes?

# Some more details

- Row player’s winnings are column player’s losses
  - thus zero-sum
  - negative numbers implies that the row player pays the column player
- Row player wants to maximize the outcome
- Column player wants to minimize the outcome
- All outcomes are assumed known by both sides.
- Neither side knows the other player’s selection before choosing a selection themselves.
  – But, row players choice depends on column players choice, and vice versa.

# Some definitions

- Each player will use a *strategy*
  - I.e., a rule for choosing an alternative from among the choices
- Assume Row player will use a *maximin* strategy (i.e., he will maximize the minimum (his worst) possible outcome).
  - The *security level* of an alternative (row) is the minimum entry in the row.
- Assume Column player will use a *minimax* strategy (i.e., he will minimize the maximum (his worst) possible outcome).
  - The *security level* of an alternative (column) is the maximum entry in the column.
- Note: crossing the street is <u>not</u> minimax
  - Assumes intelligent opponent

# Two-Player Zero-Sum game first steps

- Write down the row minimums
- The maximum of row minimums is the row player's security level ($V_I$)
  - I.e., Player I can guarantee he gets at least this amount
- Write down the column maximums
- The minimum of row maximums is the column player's security level ($V_{II}$)
  - I.e., Player II can guarantee he gets at least this amount

# More definitions and Some Results

- Fact: When the highest security-level of the rows equals the lowest security level for the columns, then the game has an *equilibrium outcome* (or "*saddle point*")
-- I.e., the game is strictly determined! Choose the row or column that maximizes or minimizes security level No reason to change! We are done!
- The value of the saddle point solution is called the *value* of the game, which is what the column player will pay to the row player.
- Mathematically:
  - Let $a_{ij}$ = payoff for row i and column j
  - $v_I = \max_i(\min_j(a_{ij}))$ -- players I's security level
  - $v_{II} = \min_j(\max_i(a_{ij}))$ -- player II's security level
  - If $v_I = v_{II}$, then there exists a saddle point
  - Game is determined, no reason to change strategy, could even announce selections in advance!
  - If $v_I < v_{II}$ we need to do more

# Can show mathematically that

- A game is said to have *perfect information* if at each stage of the play, every player is aware of all past moves by himself and others as well as all future choices that are allowed and the sides alternate moves.
- For finite games with *perfect information*, where all moves are known and people alternate moves, there exists a saddle point solution.
- Can you think of such a game?
- Why don't people use the optimal solution?

# What happens when the security levels are not equal?

- I.e., $v_I < v_{II}$
- Result, no Saddle point exists. Need to use *mix strategies*. A mix strategy uses a random device to select an action according to known probabilities.
- Fact: The value of the game, using maximin and minimax strategies is v (= expected payout from II to I) is such that $v_I < v < v_{II}$.
- Consider the game of odds-even

# Some facts about games without saddle points

- All games have a value v = expected payout from II to I. (assuming optimal playing)
- No saddle point exists <==> $v_I < v < v_{II}$ .
- Fact: given II's mixed strategy, the value of all of the rows that I plays must be equal.
- Fact: given I's mixed strategy, the value of all of the columns that II plays must be equal.
- Using this, let's derive the optimum strategy for the 2x2 Odds-Even game
- Let's do this more generally for a 2x2 game.

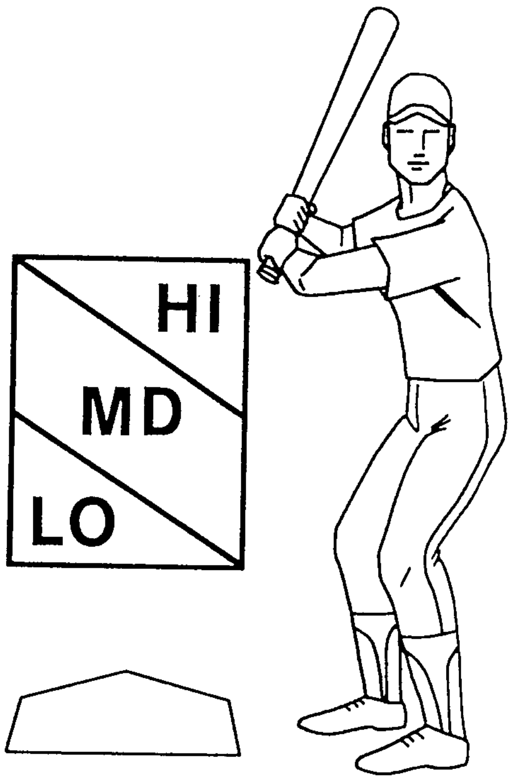
# Dominance

- Consider the following game:

- Definition: If $a_{ij} \quad a_{kj}$, for all j, then row i *dominates* row k.
  - I should never have row k in his strategy, so take it out of the game.
- Definition: If $a_{ij} \quad a_{ik}$, then column j *dominates* row k.
  - II should never have column k in his strategy, so take it out of the game.
- Always start games by taking out as many rows and columns as possible by dominance.
- Say something about stochastic dominance

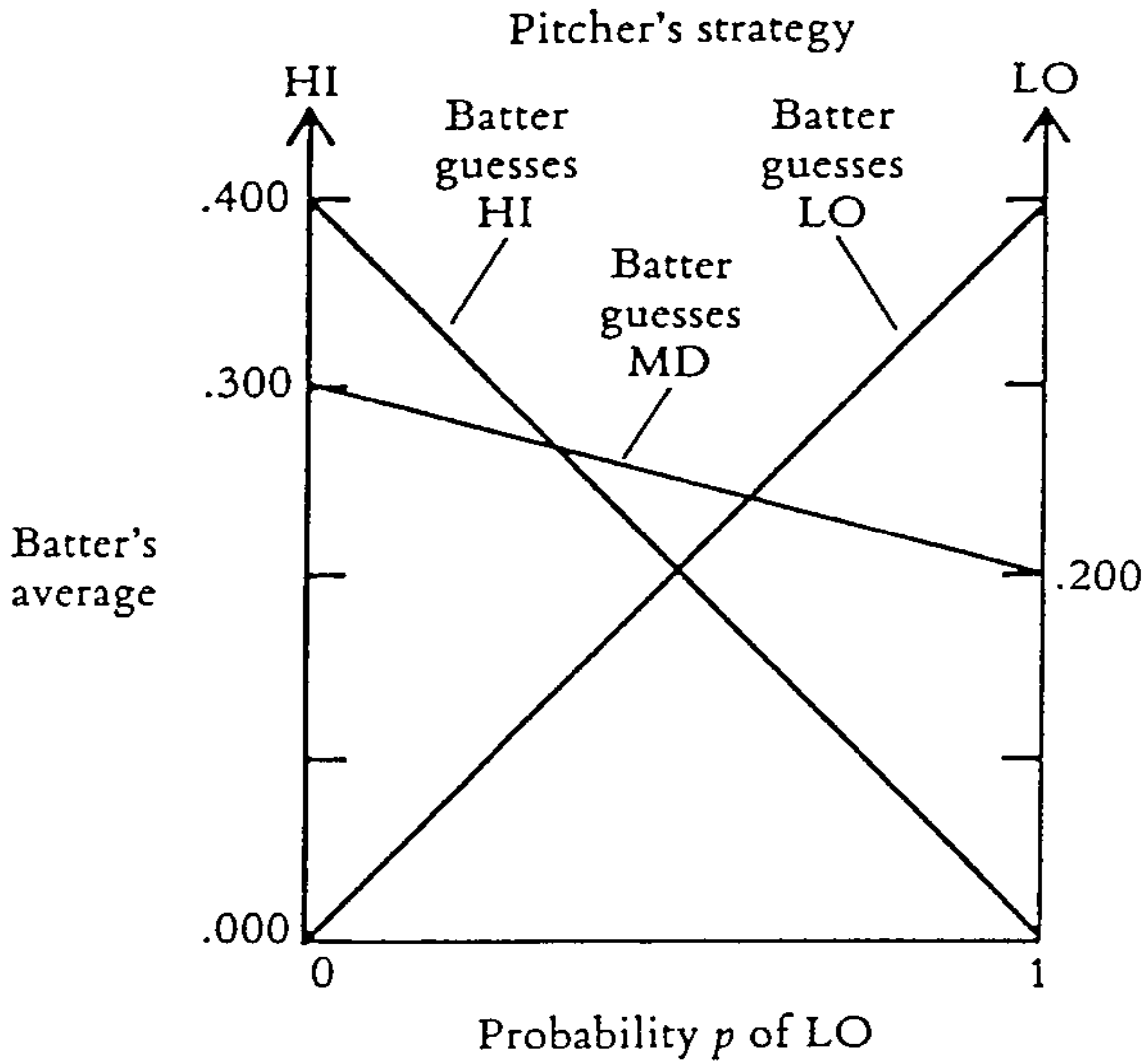
# Moving on to mx2 and 2xn Games

- Let's use the baseball example

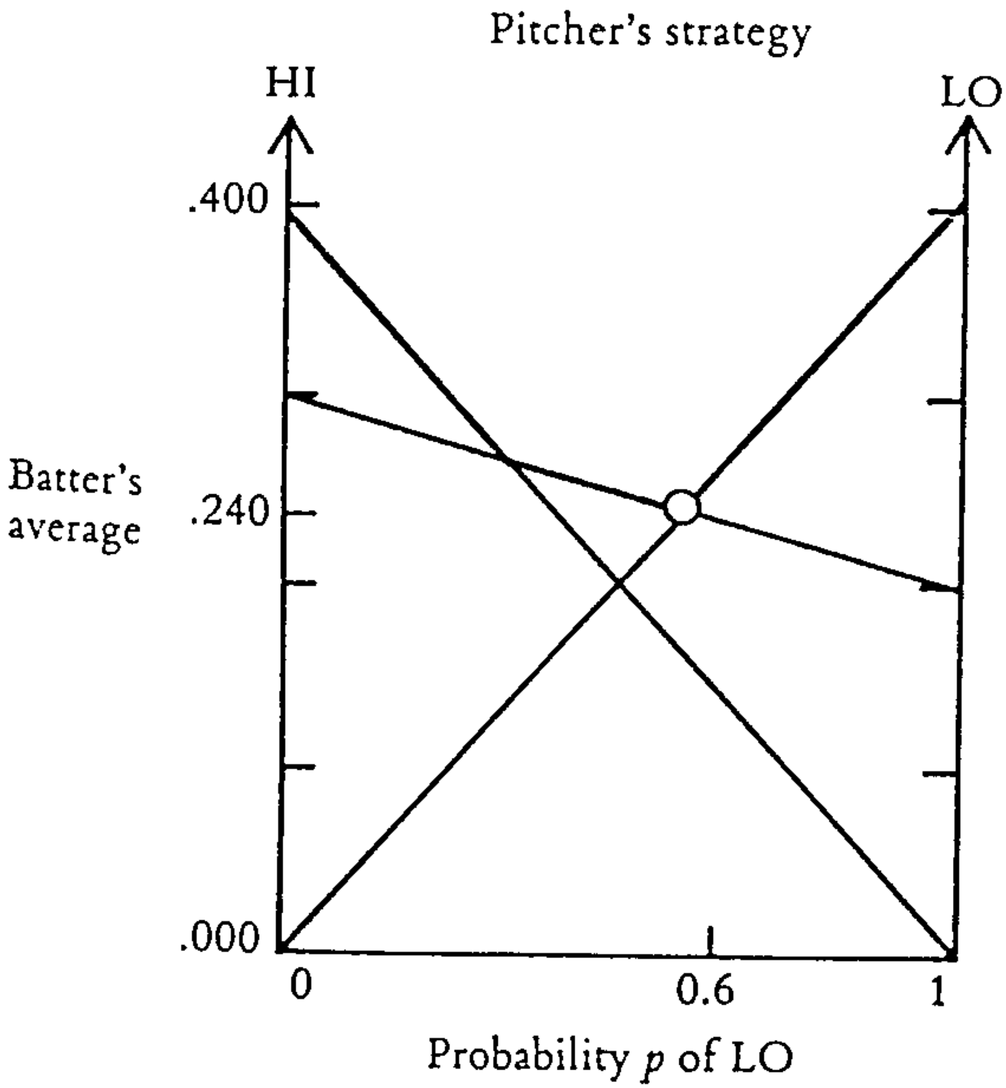



| | | *Pitcher* | | |
|---|---|---|---|---|
| | | HI | MD | LO |
| | HI | .4 | .2 | .0 |
| *Batter* | MD | .3 | .4 | .2 |
| | LO | .0 | .3 | .4 |

# Plot batter average (row payoffs ) versus the prob p that the pitcher pitches low



# Graph of the minimax solution




**<u>Note:</u>**

(1) after identify solution can solve closed formed 2x2 game

(2) For 2xn use the bottom (maximin) hull of graph

# Solving General mxn Games

- Linear Programming
  - See handout
- Fictitious play
  - See S-plus example